

## ASYMPTOTIC EXPANSION OF SMOOTH INTERVAL MAPS

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ABSTRACT. We show that several different ways to quantify the asymptotic expansion of a non-degenerate smooth interval map coincide. A consequence is an extension to multimodal maps of the remarkable result of Nowicki and Sands giving several characterizations of the Collet-Eckmann condition for unimodal maps. Combined with a result of Nowicki and Przytycki, this implies that several natural non-uniform hyperbolicity conditions are invariant under topological conjugacy. Another consequence is for the thermodynamic formalism of non-degenerate smooth interval maps: A high-temperature phase transition occurs precisely when the Topological Collet-Eckmann condition fails.

## 1. INTRODUCTION

In the last few decades, the statistical and stochastic properties of non-uniformly hyperbolic smooth maps have been extensively studied in the one-dimensional setting, see for example [BLVS03, GS09, KN92, RLS10, She11, You92] and references therein. These maps are known to be abundant, see for example [AM05, BC85, Jak81, GS11, Lyu02, Tsu01] for interval maps and [Asp04, Ree86, Smi00, GS00] for complex rational maps.

Our main result asserts that several different ways to quantify the asymptotic expansion of a non-degenerate smooth interval map coincide. This implies that several natural notions of non-uniform hyperbolicity are the same, thus extending to multimodal maps the remarkable result of Nowicki and Sands characterizing the Collet-Eckmann condition for unimodal maps, see [NS98]. Combined with a result of Nowicki and Przytycki, this implies that these non-uniform hyperbolicity conditions are invariant under topological conjugacy, see [NP98]. In particular, we obtain that for non-degenerate smooth interval maps, the property that an iterate has an exponentially mixing absolutely continuous invariant probability (acip) is invariant under topological conjugacy.

Another consequence of our main result is about the regularity and statistical properties of an arbitrary exponentially mixing acip. Combined with [Gou05, MN05, MN09, TK05, You99], our main result implies that such a measure satisfies strong statistical properties, such as the local central limit theorem and the vector-valued almost sure invariant principle. Combined with a recent result of Shen and the author in [RLS10], our main result also implies that for some  $p > 1$  the density of such a measure is in the space  $L^p(\text{Leb})$ .

Our main result has the following consequence for the thermodynamic formalism of non-degenerate smooth interval maps: A high-temperature phase transition occurs precisely when the Topological Collet-Eckmann condition fails. This last result is used in the study of the analyticity properties of the geometric pressure in [PRL12].

We proceed to describe our results more precisely. To simplify the exposition, below we state our results in a more restricted setting than what we are able to handle. For general versions, see §4 and the remarks in §6.

**1.1. Quantifying asymptotic expansion.** Let  $I$  be a compact interval and  $f : I \rightarrow I$  a smooth map. A *critical point* of  $f$  is a point of  $I$  at which the derivative of  $f$  vanishes. The map  $f$  is *non-degenerate* if it is non-injective, if the number of its critical points is finite, and if at each critical point of  $f$  some higher order derivative of  $f$  is non-zero. A non-degenerate smooth interval map is *unimodal* if it has a unique critical point.

Let  $f : I \rightarrow I$  be a non-degenerate smooth map. For an integer  $n \geq 1$ , a periodic point  $p$  of  $f$  of period  $n$  is *hyperbolic repelling* if  $|Df^n(p)| > 1$ . In this case, denote by

$$\chi_p(f) := \frac{1}{n} \ln |Df^n(p)|$$

the Lyapunov exponent of  $p$ . Similarly, for a Borel probability measure  $\nu$  on  $I$  that is invariant by  $f$  denote by

$$\chi_\nu(f) := \int \ln |Df| d\nu$$

its Lyapunov exponent.

The following is our main result. A non-degenerate smooth map  $f : I \rightarrow I$  is *topologically exact*, if for every open subset  $U$  of  $I$  there is an integer  $n \geq 1$  such that  $f^n(U) = I$ .

**Main Theorem.** *For a non-degenerate smooth map  $f : I \rightarrow I$ , the number*

$$\chi_{\inf}(f) := \inf \{ \chi_\nu(f) : \nu \text{ invariant probability measure of } f \}$$

*is equal to*

$$\chi_{\text{per}}(f) := \inf \{ \chi_p(f) : p \text{ hyperbolic repelling periodic point of } f \}.$$

*If in addition  $f$  is topologically exact, then there is  $\delta > 0$  such that for every interval  $J$  contained in  $I$  that satisfies  $|J| \leq \delta$ , we have*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \ln \max \{ |W| : W \text{ connected component of } f^{-n}(J) \} = -\chi_{\inf}(f).$$

*Moreover, for each point  $x_0$  in  $I$  we have*

$$(1.1) \quad \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \min \{ |Df^n(x)| : x \in f^{-n}(x_0) \} \leq \chi_{\inf}(f),$$

*and there is a subset  $E$  of  $I$  of zero Hausdorff dimension such that for each point  $x_0$  in  $I \setminus E$  the lim sup above is a limit and the inequality an equality.*

Except for the equality  $\chi_{\inf}(f) = \chi_{\text{per}}(f)$ , the hypothesis that  $f$  is topologically exact is necessary, see §1.6.

When restricted to the case where  $f$  is unimodal, the Main Theorem gives a quantified version of the fundamental part of [NS98, Theorem A]. In [NS98, Theorem A], property (1.1) was only considered in the case where  $x_0$  is the critical point of  $f$ ; so the assertions concerning (1.1) in the Main Theorem are new, even when restricted to the case where  $f$  is unimodal. The proof of [NS98, Theorem A] relies heavily on delicate combinatorial arguments that are specific to unimodal maps. As is, it does not extend to interval maps with several critical points. When restricted to unimodal maps, our argument is substantially simpler than that of [NS98].

When  $f$  is a complex rational map, the Main Theorem is the essence of [PRLS03, Main Theorem]. The proof in [PRLS03, Main Theorem] does not extend to interval maps, because at a key point it relies on the fact that a complex rational map is open as a map of the Riemann sphere to itself. Our argument allows us to deal with the fact that a non-degenerate smooth interval map is not an open map in general, see §1.7 for further details.

**1.2. Non-uniformly hyperbolic interval maps.** For a non-degenerate smooth interval map  $f$ , the condition  $\chi_{\inf}(f) > 0$  can be regarded as a strong form of non-uniform hyperbolicity in the sense of Pesin. A consequence of the Main Theorem is that this condition coincides with several natural non-uniform hyperbolicity conditions. To state this result more precisely, we recall some terminology.

Let  $(X, \text{dist})$  be a compact metric space,  $T : X \rightarrow X$  a continuous map and  $\nu$  a Borel probability measure that is invariant by  $T$ . Then  $\nu$  is *exponentially mixing* or *has exponential decay of correlations*, if there are constants  $C > 0$  and  $\rho$  in  $(0, 1)$  such that for every continuous function  $\varphi : X \rightarrow \mathbb{R}$  and every Lipschitz continuous function  $\psi : X \rightarrow \mathbb{R}$  we have for every integer  $n \geq 1$

$$\left| \int_X \varphi \circ f^n \cdot \psi d\nu - \int_X \varphi d\nu \int_X \psi d\nu \right| \leq C \left( \sup_X |\varphi| \right) \|\psi\|_{\text{Lip}} \rho^n,$$

where  $\|\psi\|_{\text{Lip}} := \sup_{x, x' \in X, x \neq x'} \frac{|\psi(x) - \psi(x')|}{\text{dist}(x, x')}$ .

We denote by  $\text{Leb}$  the Lebesgue measure on  $\mathbb{R}$ . For a non-degenerate smooth map  $f : I \rightarrow I$ , we use  $\text{acip}$  to refer to a Borel probability measure on  $I$  that is absolutely continuous with respect to  $\text{Leb}$  and that is invariant by  $f$ .

A non-degenerate smooth map  $f : I \rightarrow I$  satisfies the:

- *Collet-Eckmann condition*, if all the periodic points of  $f$  are hyperbolic repelling and if for every critical value  $v$  of  $f$  we have

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \ln |Df^n(v)| > 0.$$

- *Backward or Second Collet-Eckmann condition at a point  $x$  of  $I$* , if there are constants  $C > 0$  and  $\lambda > 1$ , such that for every integer  $n \geq 1$  and every point  $y$  of  $f^{-n}(x)$  we have  $|Df^n(y)| \geq C\lambda^n$ .
- *Backward or Second Collet-Eckmann condition*, if  $f$  satisfies the Backward Collet-Eckmann condition at each of its critical points.
- *Exponential Shrinking of Components condition*, if there are constants  $\delta > 0$  and  $\lambda > 1$  such that for every interval  $J$  contained in  $I$  that satisfies  $|J| \leq \delta$ , the following holds: For every integer  $n \geq 1$  and every connected component  $W$  of  $f^{-n}(J)$  we have  $|W| \leq \lambda^{-n}$ .

Finally, a non-degenerate smooth interval map  $f$  has *Uniform Hyperbolicity on Periodic Orbits*, if  $\chi_{\text{per}}(f) > 0$ .

**Corollary A.** *For a non-degenerate smooth map  $f : I \rightarrow I$  that is topologically exact, the following properties are equivalent:*

1.  $\chi_{\inf}(f) > 0$ .
2. *Uniform Hyperbolicity on Periodic Orbits* ( $\chi_{\text{per}}(f) > 0$ ).
3. *Existence of an exponentially mixing acip for  $f$ .*
4. *The map  $f$  is conjugated to a piecewise affine and expanding multimodal map by a bi-Hölder continuous function.*
5. *The map  $f$  satisfies the Exponential Shrinking of Components condition.*

6. *The map  $f$  satisfies the Backward Collet-Eckmann condition at some point of  $I$ .*

*Furthermore, these equivalent conditions are satisfied when  $f$  satisfies the Collet-Eckmann or the Backward Collet-Eckmann condition.*

When  $f$  is unimodal, the equivalence of conditions 1–5 was proved by Nowicki and Sands in [NS98, Theorem A]. They also showed, still in the case where  $f$  is unimodal, that the Collet-Eckmann and the Backward Collet-Eckmann conditions are equivalent and that each of these conditions is equivalent to conditions 1–5. In contrast, for maps with several critical points the Collet-Eckmann and the Backward Collet-Eckmann conditions are not equivalent and neither of these conditions is equivalent to conditions 1–6, see [PRLS03, §6]. When  $f$  is a complex rational map, a statement analog to Corollary A was shown by Przytycki, Smirnov, and the author in [PRLS03, Main Theorem] and in [PRL07, Theorem D], see also Remark 6.2.

Even when restricted to the case where  $f$  is unimodal, the implication  $6 \Rightarrow 1$ –5 of Corollary A is new. In fact, the main new ingredient in the proof of Corollary A is the implication  $6 \Rightarrow 5$  given by the Main Theorem. The implication  $5 \Rightarrow 4$  is also new. The rest of the implications are known, or can be easily adapted from known properties of unimodal or complex rational maps, see §6 for references.

**1.3. Exponentially mixing acip’s.** Let  $f : I \rightarrow I$  be a non-degenerate smooth map that is topologically exact and that satisfies the Exponential Shrinking of Components condition. Such a map has a unique exponentially mixing acip. In [PRL11, Theorem C], this measure is constructed using the general construction of Young in [You99].\* When a measure  $\nu$  on  $I$  can be obtained in this way, we say  $\nu$  *can be obtained through a Young tower with an exponential tail estimate*. Such a measure has several statistical properties, including the “local central limit theorem” and the “vector-valued almost sure invariant principle,” see [MN09, You99] for these results and for precisions, and [Gou05, MN05, TK05] for other statistical properties satisfied by such a measure.

On the other hand, for  $f$  as above there is  $p(f) > 1$  with the following property: For each  $p \geq 1$  the density of the unique exponentially mixing acip of  $f$  is in the space  $L^p(\text{Leb})$  if  $1 \leq p < p(f)$ , and it is not in  $L^p(\text{Leb})$  if  $p > p(f)$ . See [RLS10, Remark 2.10 and Theorem E], where a geometric characterization of  $p(f)$  is also given.†

In view of the results above, the following corollary is a direct consequence of Corollary A and of general properties of non-degenerate smooth interval maps.

**Corollary B.** *Let  $f$  be a non-degenerate smooth interval map having an exponentially mixing acip  $\nu$ . Then there is  $p > 1$  such that the density of  $\nu$  with respect to  $\text{Leb}$  is in the space  $L^p(\text{Leb})$ . Moreover,  $\nu$  can be obtained through a Young tower with an exponential tail estimate. In particular,  $\nu$  satisfies the local central limit theorem and the vector-valued almost sure invariant principle.*

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\*The proof of [PRL11, Theorem C] is written for complex rational maps and applies without change to topologically exact non-degenerate smooth interval maps. See [RLS10, Remark 2.14] for a proof written for interval maps.

†If  $f$  is unimodal and we denote its critical point by  $c$ , then  $p(f) = \ell_c/(\ell_c - 1)$ .

Alves, Freitas, Luzzatto, and Vienti showed under mild assumptions that in any dimension each polynomially mixing or stretch exponentially mixing acip can be obtained through a Young tower with the corresponding tail estimates, see [AFLV11, Theorem C]. In contrast with this last result, in Corollary B the existence of  $p > 1$  for which the density of  $\nu$  is in  $L^p(\text{Leb})$  is obtained as a consequence, and not as a hypothesis. So the following question arises naturally.

*Question 1.1.* Let  $f$  be a non-degenerate smooth interval map having an acip  $\nu$ . Does there exist  $p > 1$  such that the density of  $\nu$  with respect to  $\text{Leb}$  is in the space  $L^p(\text{Leb})$ ?

**1.4. Topological invariance.** A direct consequence of a result of Nowicki and Przytycki in [NP98], is that each of the conditions 1–6 of Corollary A is invariant under topological conjugacy. To state this result more precisely, we recall the definition of the “Topological Collet-Eckmann condition” introduced in [NP98]. Let  $f : I \rightarrow I$  be a non-degenerate smooth map that is topologically exact and fix  $r > 0$ . Given an integer  $n \geq 1$ , the *criticality of  $f^n$  at a point  $x$  of  $I$*  is the number of those  $j$  in  $\{0, \dots, n-1\}$  such that the connected component of  $f^{-(n-j)}(B(f^n(x), r))$  containing  $f^j(x)$  contains a critical point of  $f$ . Then  $f$  satisfies the *Topological Collet-Eckmann (TCE) condition*, if for some choice of  $r > 0$  there are constants  $D \geq 1$  and  $\theta$  in  $(0, 1)$ , such that the following property holds: For each point  $x$  in  $I$  the set  $G_x$  of all those integers  $m \geq 1$  for which the criticality of  $f^m$  at  $x$  is less than or equal to  $D$ , satisfies

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \#(G_x \cap \{1, \dots, n\}) \geq \theta.$$

One of the main features of the TCE condition, which is readily seen from its definition, is that it is invariant under topological conjugacy: If  $f : I \rightarrow I$  is a non-degenerate smooth map satisfying the TCE condition and  $\tilde{f} : \tilde{I} \rightarrow \tilde{I}$  is a non-degenerate smooth map that is topologically conjugated to  $f$  by a map preserving critical points, then  $\tilde{f}$  also satisfies the TCE condition. Nowicki and Przytycki showed in [NP98] that for a non-degenerate smooth interval map  $f$ , condition 5 of Corollary A implies the TCE condition and that in turn the TCE condition implies condition 2 of Corollary A. Thus, the following is a direct consequence of Corollary A and [NP98].

**Corollary C.** *For a non-degenerate smooth interval map that is topologically exact, the Topological Collet-Eckmann condition is equivalent to each of the conditions 1–6 of Corollary A. In particular, each of the conditions 1–6 of Corollary A is invariant under topological conjugacy.*

Combining [NP98] and [NS98, Theorem A], it follows that for unimodal maps the Collet-Eckmann and the Backward Collet-Eckmann conditions are both invariant under topological conjugacy. In contrast, for maps with several critical points neither of these conditions is invariant under topological conjugacy, see [PRLS03, Appendix C].

Combining Corollary C with general properties of non-degenerate smooth interval maps, we obtain the following result for maps that are not necessarily topologically exact.

**Corollary D.** *For non-degenerate smooth interval maps, the property that an iterate has an exponentially mixing acip is invariant under topological conjugacy.*

**1.5. High-temperature phase transitions.** Corollary A has a very useful application to the thermodynamic formalism of interval maps, that we proceed to describe. Let  $f : I \rightarrow I$  be a non-degenerate smooth map that is topologically exact. Denote by  $\mathcal{M}(I, f)$  the space of Borel probability measures on  $I$  that are invariant by  $f$ . For a measure  $\nu$  in  $\mathcal{M}(I, f)$ , denote by  $h_\nu(f)$  the measure theoretic entropy of  $f$  with respect to  $\nu$  and for each real number  $t$  put

$$P(t) := \sup \{h_\nu(f) - t\chi_\nu(f) : \nu \in \mathcal{M}(I, f)\}.$$

It is finite and the function  $P : \mathbb{R} \rightarrow \mathbb{R}$  so defined is the *geometric pressure function* of  $f$ . It is convex and non-increasing. It follows from the generalized Bowen formula that  $P$  has at least one zero and that its first zero is in  $(0, 1]$ , see [PRL12].

Following the usual terminology in statistical mechanics, for a real number  $t_*$  we say  $f$  has a *phase transition* at  $t_*$ , if  $P$  is not real analytic at  $t = t_*$ . In accordance with the usual interpretation of  $t > 0$  as the inverse of the temperature in statistical mechanics, if in addition  $t_* > 0$  and  $t_*$  is less than or equal to the first zero of  $P$ , then we say that  $f$  has a *high-temperature phase transition*.

The following is an easy consequence of Corollary A and of the results on the analyticity of the pressure function in [PRL11, PRL12].

**Corollary E.** *For a non-degenerate smooth interval map  $f$  that is topologically exact, the following properties are equivalent:*

1. *The map  $f$  has a high-temperature phase transition.*
2. *If we denote by  $t_0$  the first zero of  $P$ , then for every  $t \geq t_0$  we have  $P(t) = 0$ .*
3. *For every real number  $t$  we have  $P(t) \geq 0$ .*
4. *The map  $f$  does not satisfy the TCE condition.*

When  $f$  is a complex rational map, the equivalence of conditions 2–4 is part of [PRLS03, Main Theorem].<sup>‡</sup>

**1.6. Notes and references.** If the map  $f$  is not topologically exact, then by the Main Theorem we have  $\chi_{\inf}(f) = \chi_{\text{per}}(f)$ , but the remaining assertions of the Main Theorem do not hold in general. For an example, consider the logistic map with the Feigenbaum combinatorics,  $f_0$ . For this map we have  $\chi_{\inf}(f_0) = 0$ . However, if  $J$  is a small closed interval that is disjoint from the post-critical set of  $f_0$ , then the limit in the Main Theorem is strictly negative. Similarly, for every point  $x_0$  that is not in the post-critical set of  $f_0$ , the  $\limsup$  in the Main Theorem is strictly positive. This also shows that the implication  $6 \Rightarrow 1$  of Corollary A does not hold for  $f_0$ . Note also that an infinitely renormalizable map  $f$  cannot satisfy any of the conditions 1–5 of Corollary A.

See [Mih08] for further examples illustrating the difference between the Collet-Eckmann and the TCE conditions for maps with at least 2 critical points.

Luzzatto and Wang showed in [LW06] that the Collet-Eckmann condition together with a slow recurrence condition is invariant under topological conjugacy. See also [LS11] for a recent related result.

See [CRL12] and references therein for results on low-temperature phase transitions; that is, phase transitions that occur after the first zero of the geometric pressure function.

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<sup>‡</sup>In the case where  $f$  is a complex rational map, it is unclear to us if condition 1 is equivalent to 2–4.

**1.7. Strategy and organization.** To prove the Main Theorem and Corollary A we follow the structure of the proof of the analog result for complex rational maps in [PRLS03, Main Theorem]. The main difficulty is the proof that  $\chi_{\text{per}}(f) > 0$  implies the last statement of the Main Theorem, which is essentially the implication  $2 \Rightarrow 5$  of Corollary A. The proof of this fact in [PRLS03] relies in an essential way on the fact that a non-constant complex rational map is open as a map from the Riemann sphere to itself. The argument provided here allows us to deal with the fact that a multimodal map is not an open map in general. Ultimately, it relies on the fact that the boundary of a bounded interval in  $\mathbb{R}$  is reduced to 2 points.

To prove implication  $2 \Rightarrow 5$  of Corollary A we first remark that the proof of the implication  $2 \Rightarrow 6$  for rational maps in [PRLS03] applies without change to interval maps. Our main technical result is a quantified version of the implication  $6 \Rightarrow 5$  for interval maps. This is stated as Proposition 3.1, after some preliminary considerations in §2. Its proof occupies all of §3. In §4 we formulate a strengthened version of the Main Theorem, stated as the Main Theorem', and we deduce it from Proposition 3.1 and known results. In the proof we use that the Lyapunov exponent of every invariant measure supported on the Julia set is non-negative [Prz93, Theorem B]. We provide a simple proof of this fact (Proposition A.1 in Appendix A), which holds for a general continuously differentiable interval map. This result is used again in the proof of Corollary E.

The proofs of Corollaries A, D, and E are given in §6, after we prove the implication  $5 \Rightarrow 4$  of Corollary A in §5.

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## 2. PRELIMINARIES

Throughout the rest of this paper  $I$  denotes a compact interval of  $\mathbb{R}$ . We endow  $I$  with the distance  $\text{dist}$  induced by the absolute value  $|\cdot|$  on  $\mathbb{R}$ . For  $x$  in  $I$  and  $r > 0$ , we denote by  $B(x, r)$  the ball of  $I$  centered at  $x$  and of radius  $r$ . For an interval  $J$  contained in  $I$ , we denote by  $|J|$  its length and for  $\eta > 0$  we denote by  $\eta J$  the open interval of  $\mathbb{R}$  of length  $\eta|J|$  that has the same middle point as  $J$ .

Given a map  $f : I \rightarrow I$ , a subset  $J$  of  $I$  is *forward invariant* if  $f(J) = J$  and it is *completely invariant* if  $f^{-1}(J) = J$ .

**2.1. Fatou and Julia sets.** Following [MdMvS92], in this section we introduce the Fatou and Julia sets of a multimodal map and gather some of their basic properties.

A non-injective continuous map  $f : I \rightarrow I$  is *multimodal*, if there is a finite partition of  $I$  into intervals on each of which  $f$  is injective. A *turning point* of a multimodal map  $f : I \rightarrow I$  is a point in  $I$  at which  $f$  is not locally injective. The set  $\text{Sing}(f)$  is the union of  $\partial I$  and of the set of turning points of  $f$ . The *Julia set*  $J(f)$  of  $f$  is the set of all points of  $I$  such that for each of its neighborhoods  $U$  and every  $n_0 \geq 1$  there is an integer  $n \geq n_0$  such that  $f^n(U)$  intersects  $\text{Sing}(f)$ . It is a non-empty compact set that is forward invariant by  $f$ . The complement of the Julia set is called the *Fatou set* and it is denoted by  $F(f)$ . A connected component of  $F(f)$  is called *Fatou component* of  $f$ . A Fatou component  $U$  of  $f$  is *periodic* if

for some integer  $p \geq 1$  we have  $f^p(U) \subset U$ . In this case the least integer  $p$  with this property is the *period* of  $U$ .

The Julia set of a multimodal map  $f$  is forward invariant, but it is not necessarily completely invariant. However,

$$f^{-1}(J(f)) \setminus J(f) \subset \text{Sing}(f).$$

When  $J(f)$  is not completely invariant, it is possible to modify  $f$  on finitely many closed intervals contained in  $F(f)$ , to obtain a multimodal map whose Julia set is completely invariant and equal to  $J(f)$ .

**2.2. Topological exactness.** Fix a multimodal map  $f : I \rightarrow I$ . We say that  $f$  is *boundary anchored* if  $f(\partial I) \subset \partial I$  and that  $f$  is *topologically exact on  $J(f)$* , if  $J(f)$  is not reduced to a point and if every open subset of  $I$  intersecting  $J(f)$  is mapped by an iterate of  $f$  onto  $J(f)$ .

Since it is too restrictive for our applications to assume that a multimodal map is at the same time boundary anchored and topologically exact on its Julia set, we introduce the following terminology. We say that a multimodal map  $f$  is *essentially topologically exact on  $J(f)$* , if there is a compact interval  $I_0$  contained in  $I$  that contains all the critical points of  $f$  and such that the following properties hold:  $f(I_0) \subset I_0$ , the multimodal map  $f|_{I_0} : I_0 \rightarrow I_0$  is topologically exact on  $J(f|_{I_0})$ , and  $\bigcup_{n=0}^{+\infty} f^{-n}(I_0)$  contains an interval whose closure contains  $J(f)$ .

**2.3. Differentiable interval maps.** Fix a differentiable map  $f : I \rightarrow I$ .

A *critical point* of  $f$  is a point at which the derivative of  $f$  vanishes. We denote by  $\text{Crit}(f)$  the set of critical points of  $f$ . A *critical value* of  $f$  is the image by  $f$  of a critical point. If  $f$  is in addition a multimodal map, then we put

$$\text{Crit}'(f) := \text{Crit}(f) \cap J(f).$$

Let  $J$  be an interval contained in  $I$  and let  $n \geq 1$  be an integer. Then each connected component of  $f^{-n}(J)$  is a *pull-back of  $J$  of order  $n$* , or just a *pull-back of  $J$* . If in addition  $f^n : W \rightarrow J$  is a diffeomorphism, then the pull-back  $W$  is *diffeomorphic*. Note that if  $f$  is boundary anchored and  $W$  is a pull-back of  $J$  of order  $n$ , then  $f^n(\partial W) \subset \partial J$ .

Let  $J$  be an interval contained in  $I$ , let  $n \geq 1$  be an integer, and let  $W$  be a pull-back of  $J$  by  $f^n$ . We say  $W$  is a *child of  $J$* ,<sup>§</sup> if  $W$  contains a unique critical point  $c$  of  $f$  in  $J(f)$  and if there is  $s$  in  $\{0, \dots, n-1\}$  such that  $f^s(c)$  belongs to  $\text{Crit}(f)$  and such that the following properties hold:

1. Either  $s = n-1$  or the pull-back of  $J$  by  $f^{n-s-1}$  containing  $f^{s+1}(c)$  is diffeomorphic.
2. For each  $s'$  in  $\{0, \dots, s\}$  the pull-back of  $J$  by  $f^{n-s'}$  containing  $f^{s'}(c)$  is either disjoint from  $\text{Crit}(f)$  or  $f^{s'}(c)$  belongs to  $\text{Crit}(f)$  and then  $f^{s'}(c)$  is the unique critical point of  $f$  contained in this set.

**2.4. Interval maps of class  $C^3$  with non-flat critical points.** A non-injective interval map  $f : I \rightarrow I$  is of class  $C^3$  with non-flat critical points if:

- The map  $f$  is of class  $C^3$  outside  $\text{Crit}(f)$ .

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<sup>§</sup>This definition is a variant of the usual definition of “child.” It is adapted to deal with the case where  $f$  has a critical connection.



- For each critical point  $c$  of  $f$  there exists a number  $\ell_c > 1$  and diffeomorphisms  $\phi$  and  $\psi$  of  $\mathbb{R}$  of class  $C^3$ , such that  $\phi(c) = \psi(f(c)) = 0$  and such that on a neighborhood of  $c$  on  $I$  we have,

$$|\psi \circ f| = \pm |\phi|^{\ell_c}.$$

The number  $\ell_c$  is the *order of  $f$  at  $c$* .

Denote by  $\mathcal{A}$  the collection of interval maps of class  $C^3$  with non-flat critical points, whose Julia set is completely invariant. Note that every interval map in  $\mathcal{A}$  is a continuously differentiable multimodal map. On the other hand, each smooth non-degenerate interval map whose Julia set is completely invariant is contained in  $\mathcal{A}$ . In particular, if  $f : I \rightarrow I$  is a non-degenerate smooth map that is topologically exact, then  $J(f) = I$  and  $f$  is in  $\mathcal{A}$ .

We use the important fact that each map in  $\mathcal{A}$  every Fatou component is mapped to a periodic Fatou component under forward iteration, see [MdMvS92, Theorem A']. We also use the fact that each interval map in  $\mathcal{A}$  has at most a finite number of periodic Fatou components, see [MdMvS92, §1].

The following version of the Koebe principle follows from [vSV04, Theorem C (2)(ii)]. As for non-degenerate smooth interval maps, a periodic point  $p$  of period  $n$  of a map  $f$  in  $\mathcal{A}$  is *hyperbolic repelling* if  $|Df^n(p)| > 1$ .

**Lemma 2.1** (Koebe principle). *Let  $f : I \rightarrow I$  be an interval map in  $\mathcal{A}$  all whose periodic points in  $J(f)$  are hyperbolic repelling. Then there is  $\delta_0 > 0$  such that for every  $K > 1$  there is  $\varepsilon$  in  $(0, 1)$  such that the following property holds. Let  $J$  be an interval contained in  $I$  that intersects  $J(f)$  and satisfies  $|J| \leq \delta_0$ . Moreover, let  $n \geq 1$  be an integer and  $W$  a diffeomorphic pull-back of  $J$  by  $f^n$ . Then for every  $x$  and  $x'$  in the unique pull-back of  $\varepsilon J$  by  $f^n$  contained in  $W$  we have*

$$K^{-1} \leq |Df^n(x)|/|Df^n(x')| \leq K.$$

The following general fact is used in the proof of the Main Theorem' in §4.

**Fact 2.2.** *If  $f$  is an interval map in  $\mathcal{A}$  that is topologically exact on  $J(f)$ , then  $J(f)$  contains a uniformly expanding set whose topological entropy is strictly positive. In particular, the Hausdorff dimension of  $J(f)$  is strictly positive.*

The following lemma is standard, see for example [RL12, Lemmas A.2 and A.3].

**Lemma 2.3.** *Let  $f : I \rightarrow I$  be a multimodal map in  $\mathcal{A}$  having all of its periodic points in  $J(f)$  hyperbolic repelling. Then the following properties hold.*

1. *There is  $\delta_1 > 0$  such that for every  $x$  in  $J(f)$ , every integer  $n \geq 1$ , every pull-back  $W$  of  $B(x, \delta_1)$  by  $f^n$  intersects  $J(f)$ , contains at most 1 critical point of  $f$ , and is disjoint from  $(\text{Crit}(f) \cup \partial I) \setminus J(f)$ .*
2. *If in addition  $f$  is essentially topologically exact on  $J(f)$ , then for every  $\kappa > 0$  there is  $\delta_2 > 0$  such that for every  $x$  in  $J(f)$ , every integer  $n \geq 1$ , and every pull-back  $W$  of  $B(x, \delta_2)$  by  $f^n$ , we have  $|W| \leq \kappa$ .*

### 3. EXPONENTIAL SHRINKING OF COMPONENTS

The purpose of this section is to prove the following proposition. It is the key step in the proof of the Main Theorem, which is given in the next section.

**Proposition 3.1.** *Let  $f : I \rightarrow I$  be a map in  $\mathcal{A}$  that is topologically exact on  $J(f)$ . Suppose there is a point  $x_0$  of  $J(f)$  and constants  $C > 0$  and  $\lambda > 1$  such that for every integer  $n \geq 1$  and every point  $x$  in  $f^{-n}(x_0)$  we have*

$$|Df^n(x)| \geq C\lambda^n.$$

*Then every periodic point of  $f$  in  $J(f)$  is hyperbolic repelling and for every  $\lambda_0$  in  $(1, \lambda)$  there is a constant  $\delta_3 > 0$  such that the following property holds. Let  $J$  be an interval contained in  $I$  that intersects  $J(f)$  and satisfies  $|J| \leq \delta_3$ . If  $J(f)$  is not an interval, then assume that  $J$  is not a neighborhood of a periodic point in the boundary of a Fatou component of  $f$ .<sup>¶</sup> Then for every integer  $n \geq 1$  and every pull-back  $W$  of  $J$  by  $f^n$ , we have*

$$(3.1) \quad |W| \leq \lambda_0^{-n}.$$

The proof of this proposition is at the end of this section. It is based on several lemmas.

In this section, a critical point  $c$  of a map  $f$  in  $\mathcal{A}$  is *exposed*, if for every integer  $j \geq 1$  the point  $f^j(c)$  is not a critical point of  $f$ . Given  $c$  in  $\text{Crit}'(f)$ , let  $s \geq 0$  be the largest integer such that  $f^s(c)$  is in  $\text{Crit}(f)$  and put

$$\widehat{\ell}_c := \prod_{\substack{j \in \{0, \dots, s\} \\ f^j(c) \in \text{Crit}(f)}} \ell_{f^j(c)} \text{ and } \widehat{\ell}_{\max} := \max \left\{ \widehat{\ell}_c : c \in \text{Crit}'(f) \right\}.$$

**Lemma 3.2.** *Let  $f : I \rightarrow I$  be a boundary anchored interval map in  $\mathcal{A}$  that is essentially topologically exact on  $J(f)$  and such that all of its periodic points in  $J(f)$  are hyperbolic repelling. Then there are  $\delta_4 > 0$  and  $C_1 > 1$  such that for every interval  $J$  that intersects  $J(f)$  and satisfies  $|J| \leq \delta_4$  and  $C_1 J \subset I$ , the following property holds: For every integer  $n \geq 1$  and every pull-back  $W$  of  $J$  by  $f^n$  such that the pull-back of  $C_1 J$  by  $f^n$  containing  $W$  is a child of  $C_1 J$ , we have*

$$|W| \leq 6\widehat{\ell}_{\max}|J| \max \{ |Df^n(a)| : a \in \partial W \}^{-1}.$$

*Proof.* Let  $\delta_0 > 0$  and  $\varepsilon$  in  $(0, 1)$  be given by Lemma 2.1 with  $K = 2$ . Since the critical points of  $f$  are non-flat, there is  $\delta_* > 0$  so that for each  $c$  in  $\text{Crit}'(f)$ , each integer  $s \geq 0$  such that  $f^s(c)$  is in  $\text{Crit}'(f)$ , and each interval  $W$  contained in  $B(c, \delta_*)$  we have

$$|W| \max \{ |Df^{s+1}(a)| : a \in \partial W \} \leq 3\widehat{\ell}_c |f^{s+1}(W)|.$$

Let  $\delta_2 > 0$  be given by part 2 of Lemma 2.3 with  $\kappa = \delta_*$ .

We prove the lemma with  $\delta_4 = \varepsilon \min\{\delta_2, \delta_0\}$  and  $C_1 = \varepsilon^{-1}$ . To do this, let  $J$  be an interval contained in  $I$  that intersects  $J(f)$  and satisfies  $|J| \leq \delta_2$  and  $\widehat{J} := \varepsilon^{-1}J \subset I$ , let  $n \geq 1$  be an integer and let  $W$  be a pull-back of  $J$  by  $f^n$  such that the pull-back  $\widehat{W}$  of  $\widehat{J}$  by  $f^n$  containing  $W$  is a child of  $\widehat{J}$ . Let  $c$  be the unique critical point of  $f$  contained in  $\widehat{W}$  and let  $s$  be the largest element of  $\{0, \dots, n-1\}$  such that  $f^s(c)$  is in  $\text{Crit}(f)$ . So either  $s = n-1$  or the pull-back  $\widehat{W}'$  of  $\widehat{J}$  by  $f^{n-s-1}$  containing  $f^{s+1}(W)$  is diffeomorphic. Then the Koebe principle (Lemma 2.1) implies that, if we denote by  $W'$  the pull-back of  $J$  by  $f^{n-s-1}$  containing  $f^{s+1}(W)$ , then

$$|W'| \leq 2|J| \max \{ |Df^{n-s-1}(a')| : a' \in \partial W' \}^{-1}.$$

<sup>¶</sup>There is an example showing that this hypothesis is necessary, see [RL12, Proposition A]. However, a qualitative result holds when this hypothesis is not satisfied, see [RL12, Theorem B].

On the other hand, by part 2 of Lemma 2.3 we have  $W \subset \widehat{W} \subset B(c, \delta_*)$ , so by our choice of  $\delta_*$  we have

$$\begin{aligned} |W| &\leq 3\widehat{\ell}_c |f^{s+1}(W)| \max \{ |Df^{s+1}(a)| : a \in \partial W \}^{-1} \\ &\leq 3\widehat{\ell}_{\max} |\widehat{W}'| \max \{ |Df^{s+1}(a)| : a \in \partial W \}^{-1}. \end{aligned}$$

The desired inequality is obtained by combining the last two displayed inequalities.  $\square$

**Lemma 3.3.** *Let  $f : I \rightarrow I$  be a boundary anchored interval map in  $\mathcal{A}$  that is essentially topologically exact on  $J(f)$  and such that all of its periodic points in  $J(f)$  are hyperbolic repelling. Suppose that none of the boundary points of  $I$  is a critical point of  $f$  and let  $C_1 > 1$  be the constant given by Lemma 3.2. Then, for every  $\eta > 1$  there is a constant  $\delta(\eta) > 0$  such that for every interval  $\widehat{J}$  that intersects  $J(f)$  and satisfies  $|\widehat{J}| \leq \delta(\eta)$  and  $C_1 \widehat{J} \subset I$ , the following properties hold for every integer  $n \geq 1$  and every pull-back  $\widehat{W}$  of  $\widehat{J}$  by  $f^n$ :*

1. *For every interval  $J$  contained in  $\widehat{J}$ , the number of pull-backs of  $J$  by  $f^n$  contained in  $\widehat{W}$  is bounded from above by  $2\eta^n$ .*
2.  *$|\widehat{W}| \leq 12\widehat{\ell}_{\max}\eta^n |\widehat{J}| \max \{ |Df^n(a)| : a \in \partial \widehat{W} \}^{-1}$ .*

*Proof.* Let  $\delta_0 > 0$  and  $\varepsilon$  in  $(0, 1)$  be given by Lemma 2.1 with  $K = 2$ , let  $\delta_1 > 0$  be given by part 1 of Lemma 2.3, and let  $\delta_4 > 0$  and  $C_1 > 1$  be given by Lemma 3.2. Enlarging  $C_1$  if necessary we assume  $C_1 \geq \varepsilon^{-1}$ . On the other hand, let  $L \geq 1$  be a sufficiently large integer such that  $\eta^L > 6\widehat{\ell}_{\max}$  and let  $\delta_* > 0$  be sufficiently small so that for every exposed critical point  $c$  of  $f$  and every  $j$  in  $\{0, \dots, L\}$ , the point  $f^j(c)$  is not in  $B(\text{Crit}(f), \delta_*)$ . Finally, let  $\delta_2 > 0$  be given by part 2 of Lemma 2.3 with

$$\kappa = C_1^{-1} \min \{ \delta_0, \delta_1, \delta_4, \delta_*, \text{dist}(\text{Crit}(f), \partial I) \}.$$

We prove the lemma with  $\delta(\eta) = \delta_2$ . To do this, let  $\widehat{J}$  be an interval that intersects  $J(f)$  and satisfies  $|\widehat{J}| \leq \delta_2$  and  $C_1 \widehat{J} \subset I$ , let  $n \geq 1$  be an integer, and let  $\widehat{W}$  be a pull-back of  $\widehat{J}$  by  $f^n$ . Put  $m_0 := n$  and  $\widehat{W}_0 := \widehat{J}$  and define inductively an integer  $k \geq 0$  and integers

$$m_0 > m_1 > \dots > m_k \geq 0,$$

such that for each  $t$  in  $\{1, \dots, k\}$  the pull-back  $\widehat{W}_t$  of  $\widehat{J}$  by  $f^{n-m_t}$  containing  $f^{m_t}(\widehat{W})$  is contained in  $B(\text{Crit}(f), \kappa)$ . Note that by our choice of  $\kappa$  this last property implies that  $C_1 \widehat{W}_t \subset I$ . Recalling that  $m_0 = n$ , let  $t \geq 0$  be an integer such that  $m_t$  is already defined. If  $m_t = 0$ , or if the pull-back of  $C_1 \widehat{W}_t$  by  $f^{m_t}$  containing  $\widehat{W}$  is diffeomorphic, then put  $k = t$  and stop. Otherwise, define  $m'_{t+1}$  as the largest integer  $m$  in  $\{0, \dots, m_t - 1\}$  such that the pull-back  $\widehat{W}'_{t+1}$  of  $C_1 \widehat{W}_t$  by  $f^{m_t-m}$  containing  $f^m(\widehat{W})$  is not diffeomorphic. In view of part 1 of Lemma 2.3, it follows that the set  $\widehat{W}'_{t+1}$  contains a unique critical point and that this critical point is exposed and belongs to  $J(f)$ . Moreover,  $\widehat{W}'_{t+1}$  is a child of  $C_1 \widehat{W}_t$ . Define  $m_{t+1}$  as the smallest integer  $m$  in  $\{0, \dots, m'_{t+1}\}$  such that the pull-back  $W_*$  of  $C_1 \widehat{W}_t$  by  $f^{m_t-m}$  containing  $f^m(\widehat{W})$  is a child of  $C_1 \widehat{W}_t$ . Clearly,  $\widehat{W}_{t+1} \subset W_* \subset B(\text{Crit}(f), \kappa)$ .

Note that if  $k = 0$ , then the pull-back of  $C_j \widehat{J}$  by  $f^n$  containing  $\widehat{W}$  is diffeomorphic; in particular  $f^n : \widehat{W} \rightarrow \widehat{J}$  is diffeomorphic. On the other hand, note that by

definition of  $L$ , for every  $t$  in  $\{2, \dots, k\}$  we have

$$m_{t-1} - m_t \geq m_{t-1} - m'_t \geq L.$$

To prove part 1 of the lemma, observe that if  $k = 0$ , then  $f^n : \widehat{W} \rightarrow \widehat{J}$  is a diffeomorphism and the desired assertion is trivially true. Suppose  $k \geq 1$  and let  $J$  be an interval contained in  $\widehat{J}$ . It follows from the definitions that for every  $t$  in  $\{1, \dots, k\}$  the map  $f^{m_{t-1}-m_t}$  has at most one critical point in  $f^{m_t}(\widehat{W})$ . Furthermore, an induction argument in  $t$  shows that there are at most  $2^t$  pull-backs of  $J$  by  $f^{n-m_t}$  contained in the pull-back of  $\widehat{J}$  containing  $f^{m_t}(\widehat{W})$ . Since

$$2^k \leq 2\eta^{(k-1)L} \leq 2\eta^{m_1-m_k} \leq 2\eta^n,$$

the last assertion with  $t = k$  proves part 1 of the lemma in the case where  $m_k = 0$ . If  $m_k \geq 1$ , then it follows from the definitions that the pull-back of  $C_1\widehat{W}_k$  by  $f^{m_k}$  containing  $\widehat{W}$  is diffeomorphic. So the number of pull-backs of  $J$  by  $f^n$  contained in  $\widehat{W}$  is also bounded from above by  $2\eta^n$ . This completes the proof of part 1 of the lemma.

To prove part 2, suppose first  $k = 0$ . Then the pull-back of  $C_1\widehat{J}$  by  $f^n$  containing  $\widehat{W}$  is diffeomorphic and the desired inequality follows from the Koebe principle (Lemma 2.1) with  $12\widehat{\ell_{\max}}\eta^n$  replaced by 2. Suppose  $k \geq 1$  and observe that by Lemma 3.2 for each  $t$  in  $\{1, \dots, k\}$  we have

$$|\widehat{W}_t| \leq 6\widehat{\ell_{\max}}|\widehat{W}_{t-1}| \max \left\{ |Df^{m_{t-1}-m_t}(a)| : a \in \partial\widehat{W}_t \right\}^{-1}.$$

By an induction argument we obtain,

$$|\widehat{W}_k| \leq (6\widehat{\ell_{\max}})^k |\widehat{J}| \max \left\{ |Df^{n-m_k}(a')| : a' \in \partial\widehat{W}_k \right\}^{-1}.$$

Using

$$(6\widehat{\ell_{\max}})^{k-1} < \eta^{(k-1)L} \leq \eta^{m_1-m_k} \leq \eta^n,$$

we obtain

$$|\widehat{W}_k| \leq 6\widehat{\ell_{\max}}\eta^n \max \left\{ |Df^{n-m_k}(a) : a \in \partial\widehat{W}_k \right\}^{-1}.$$

This proves part 2 of the lemma in the case where  $m_k = 0$ . If  $m_k \geq 1$ , then the pull-back of  $C_1\widehat{W}_k$  by  $f^{m_k}$  containing  $\widehat{W}$  is diffeomorphic and by the Koebe principle (Lemma 2.1) we obtain

$$\begin{aligned} |\widehat{W}| &\leq 2|\widehat{W}_k| \max \left\{ |Df^{m_k}(a)| : a \in \partial\widehat{W} \right\}^{-1} \\ &\leq 12\widehat{\ell_{\max}}|\widehat{J}| \max \left\{ |Df^n(a)| : a \in \partial\widehat{W} \right\}^{-1}. \end{aligned}$$

This completes the proof of part 2 and of the lemma.  $\square$

The following lemma is more general than what we need for the proof of Proposition 3.1. It is used again in the proof of the Main Theorem in the next section.

**Lemma 3.4.** *Let  $f : I \rightarrow I$  be an interval map in  $\mathcal{A}$  that is topologically exact on  $J(f)$  and put*

$$\chi_{\text{per}}^0(f) := \inf \{ \chi_p(f) : p \text{ periodic point of } f \text{ in } J(f) \}.$$

Then for every interval  $J$  contained in  $I$  that intersects  $J(f)$  we have

$$(3.2) \quad \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \max \{ |W| : W \text{ connected component of } f^{-n}(J) \} \geq -\chi_{\text{per}}^0(f)$$

and for every point  $x_0$  of  $J(f)$  we have

$$(3.3) \quad \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \min \{ |Df^n(x)| : x \in f^{-n}(x_0) \} \leq \chi_{\text{per}}^0(f).$$

*Proof.* Let  $\ell \geq 1$  be an integer and let  $p$  be a periodic point of  $f$  of period  $\ell$  in  $J(f)$ .

Suppose first  $p$  is hyperbolic repelling. Then there is  $\delta > 0$  and a uniformly contracting inverse branch  $\phi$  of  $f^\ell$  that is defined on  $B(p, \delta)$  and fixes  $p$ . It follows that  $\phi(\overline{B(p, \delta)}) \subset B(p, \delta)$  and that there is  $K > 1$  such that for every integer  $k \geq 1$  the distortion of  $\phi^k$  on  $B(p, \delta)$  is bounded by  $K$ . On the other hand, the hypothesis that  $f$  is topologically exact on  $J(f)$  implies that there is an integer  $m \geq 1$  such that the intersection of  $f^{-m}(J)$  and  $B(p, \delta)$  contains an interval  $J'$  and such that there is a point  $x'_0$  in  $f^{-m}(x_0)$  contained in  $B(p, \delta)$ . Then we have

$$(3.4) \quad \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \max \{ |W| : W \text{ connected component of } f^{-n}(J) \} \geq \liminf_{k \rightarrow +\infty} \frac{1}{k\ell} \ln |\phi^k(J')| = -\chi_p(f)$$

and

$$(3.5) \quad \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \min \{ |Df^n(x)| : x \in f^{-n}(x_0) \} \leq - \lim_{k \rightarrow +\infty} \frac{1}{k\ell} \ln |D\phi^k(x'_0)| = \chi_p(f).$$

Since  $p$  is an arbitrary hyperbolic repelling periodic point, this proves (3.2) and (3.3).

It remains to consider the case where  $p$  is not hyperbolic repelling, so that  $Df^{2\ell}(p) = 1$ . Without loss of generality we assume that for every  $\delta > 0$  the interval  $(p, p + \delta)$  intersects  $J(f)$ . Let  $\eta > 1$  be given and let  $\delta > 0$  be sufficiently small so there is an inverse branch  $\phi$  of  $f^{2\ell}$  that is defined on  $B(p, \delta)$ , that fixes  $p$ , and that is strictly increasing on  $(p, p + \delta)$ . Reducing  $\delta$  if necessary we assume we have  $|Df| < \eta$  on  $B(p, \delta)$ . As in the previous case there is an integer  $m \geq 1$  such that the intersection of  $f^{-m}(J)$  and  $(p, p + \delta)$  contains an interval  $J'$  and such that there is a point  $x'_0$  in  $f^{-m}(x_0)$  contained in  $(p, p + \delta)$ . Then we have (3.4) and (3.5) with  $\chi_p(f)$  replaced by  $\varepsilon$ . Since  $\varepsilon > 0$  is arbitrary, these inequalities hold with  $\chi_p(f) = 0$ . The proof of the lemma is thus completed.  $\square$

*Proof of Proposition 3.1.* By Lemma 3.4 all the periodic points of  $f$  in  $J(f)$  are hyperbolic repelling. In view of part 2 Lemma 2.3, it is enough to show that there is a constant  $C_0 > 0$  such that the proposition holds with the right hand side of (3.1) replaced by  $C_0 \lambda_0^{-n}$ .

Let  $\tilde{I}$  be equal to  $I$  if  $J(f) = I$ . Otherwise, for each periodic point  $y$  in the boundary of a Fatou component  $U$  of  $f$ , let  $y'$  be a point in  $U$ , let  $U_y$  be the open interval bounded by  $y$  and  $y'$ , and put

$$\tilde{I} := I \setminus \bigcup_y U_y,$$

where the union runs through all the periodic points of in the boundary of a Fatou component of  $f$ . In all the cases  $\tilde{I}$  is a finite union of closed intervals. In part 1 below we show that for every  $y$  in  $J(f)$  there is a constant  $C_y > 0$  and an interval  $J_y$  contained in  $\tilde{I}$  that is a neighborhood of  $y$  in  $\tilde{I}$  and such that for every integer  $n \geq 1$  and every pull-back  $W$  of  $J_y$  by  $f^n$  we have

$$|W| \leq C_y \lambda_0^{-n}.$$

Since  $J(f)$  is compact, this implies the proposition, except in the case where  $J(f)$  is an interval having a boundary point in the interior of  $I$  that is a periodic point of  $f$ . This last case is treated in part 2.

Let  $\hat{I}$  be a compact interval containing  $I$  in its interior and let  $\hat{f} : \hat{I} \rightarrow \hat{I}$  be an extension of  $f$  in  $\mathcal{A}$  that is boundary anchored, such that all the critical points of  $\hat{f}$  are contained in  $I$ , and such that  $\bigcup_{n=0}^{+\infty} \hat{f}^{-n}(I)$  contains an interval whose closure contains  $J(\hat{f})$ . Note in particular that  $\hat{f}$  is essentially topologically exact on  $J(\hat{f})$ . Without loss of generality we assume that all the periodic points of  $\hat{f}$  in  $J(\hat{f})$  are hyperbolic repelling. Put  $\eta := (\lambda/\lambda_0)^{1/2}$  and let  $\delta_* > 0$  be the constant  $\delta(\eta)$  given by Lemma 3.3 with  $f$  replaced by  $\hat{f}$ . Moreover, let  $C_1 > 1$  be the constant given by Lemma 3.2. Reducing  $\delta_*$  if necessary we assume

$$\delta_* < C_1^{-1} \text{dist}(I, \partial \hat{I}).$$

Note that this implies that for every interval  $J$  intersecting  $I$  and satisfying  $|J| \leq \delta_*$ , we have  $C_1 J \subset \hat{I}$ .

**1.** Suppose first  $y$  is not a boundary point of a Fatou component of  $f$  of length greater than or equal to  $\delta_*/2$ . Since  $f$  is topologically exact on  $J(f)$ , we can find an integer  $n_0 \geq 1$  and points  $x$  and  $x'$  in  $f^{-n_0}(x_0)$  such that

$$x < y < x' \text{ and } |x - x'| < \delta_*.$$

Then the desired assertion follows with

$$J_y = (x, x') \text{ and } C_y = 12\ell_{\max} C^{-1} \delta_*,$$

by part 2 of Lemma 3.3 with  $f$  replaced by  $\hat{f}$  and with  $\hat{J} = (x, x')$ .

Suppose  $y$  is a boundary point of a Fatou component of  $f$  and that  $y$  is not periodic. Then there is an integer  $N \geq 1$  such that every point in  $f^{-N}(y)$  is either not in the boundary of a Fatou component or in the boundary of a Fatou component of length strictly smaller than  $\delta_*/2$ . Then the desired assertion follows from the previous case.

It remains to consider the case where  $y$  is a periodic point in the boundary of a Fatou component of length greater than or equal to  $\delta_*/2$ . Let  $\pi \geq 1$  be the period of  $y$  and let  $\delta$  in  $(0, \delta_*/2)$  be sufficiently small so that there is an inverse  $\phi$  of  $\hat{f}^\pi$  defined on  $B(y, \delta)$ , fixing  $y$  and such that  $\phi(\overline{B(y, \delta)}) \subset B(y, \delta)$ . Since  $\delta < \delta_*/2$  and  $y$  is a boundary point of a Fatou component of  $f$  of length greater than or equal to  $\delta_*/2$ , it follows that  $\phi$  is strictly increasing. Let  $n_0 \geq 1$  be a sufficiently large integer so that  $f^{-n_0}(x_0)$  intersects  $B(y, \delta)$  and let  $y_0$  be a point of  $f^{-n_0}(x_0)$  in  $B(y, \delta)$ . For each integer  $j \geq 1$  put  $y_j := \phi^j(y_0)$  and let  $K_{j-1}$  be the closed interval bounded by  $y_{j-1}$  and  $y_j$ . Note that the intervals  $(K_j)_{j=0}^{+\infty}$  have pairwise disjoint interiors and that the closure of their union is equal to the closed interval  $J_y$  bounded by  $y$  and  $y_0$ . Clearly  $J_y$  is a neighborhood of  $y$  in  $\tilde{I}$ . On the other hand, for each integer  $j \geq 1$  the interval  $K_j$  is equal to  $\phi^j(K_0)$  and it is a pull-back of  $K_0$

by  $\widehat{f}^{\pi j}$ . So, part 2 of Lemma 3.3 with  $\widehat{J} = K_0$ , with  $f$  replaced by  $\widehat{f}$ , and with  $n$  replaced by  $n + \pi j$ , shows that for every pull-back  $W$  of  $K_j$  by  $\widehat{f}^n$  we have

$$\begin{aligned} |W| &\leq 12\widehat{\ell}_{\max}\eta^{n+j\pi}|K_0|\max\left\{|D\widehat{f}^{n+j\pi}(a)| : a \in \partial W\right\} \\ &\leq 12\widehat{\ell}_{\max}\eta^{n+j\pi}\delta_*C^{-1}\lambda^{-(n+j\pi+n_0)}\min\left\{|D\widehat{f}^{n_0}(y_0)|, |D\widehat{f}^{n_0+\pi}(y_1)|\right\}. \end{aligned}$$

On the other hand, by part 1 of Lemma 3.3 with  $f$  replaced by  $\widehat{f}$  and with  $\widehat{J} = J_y$  and  $J = K_j$ , every pull-back  $\widehat{W}$  of  $J_y$  by  $f^n$  contains at most  $2\eta^n$  pull-backs of  $K_j$  by  $f^n$ . So, letting

$$C' := 12\widehat{\ell}_{\max}\delta_*C^{-1}\lambda^{-n_0}\min\left\{|D\widehat{f}^{n_0}(y_0)|, |D\widehat{f}^{n_0+\pi}(y_1)|\right\}$$

and using the definition of  $\eta$  we obtain

$$|\widehat{W} \cap \widehat{f}^{-n}(K_j)| \leq 2\eta^n C' \eta^{n+j\pi} \lambda^{-(n+j\pi)} \leq 2C' \lambda_0^{-(n+j\pi)}.$$

Since  $J_y$  is the closure of  $\bigcup_{j \geq 0} K_j$ , summing over  $j$  we get

$$|\widehat{W}| \leq 2C' \sum_{j=0}^{+\infty} \lambda_0^{-(n+j\pi)} = 2C'(1 - \lambda_0^{-\pi})^{-1} \lambda_0^{-n}.$$

This proves the desired assertion with  $C_y = 2C'(1 - \lambda_0^{-\pi})^{-1}$ .

**2.** Suppose that  $J(f)$  is an interval having a boundary point  $y$  in the interior of  $I$  that is a periodic point of  $f$ . In view of part 1, it is enough to show that for each such point  $y$  there are  $\delta > 0$  and  $C > 0$  such that for every integer  $n \geq 1$  and every pull-back  $W$  of  $B(y, \delta)$  by  $f^n$ , we have  $|W| \leq C\lambda_0^{-n}$ . By part 1 there are  $\delta > 0$  and  $C > 0$  such that this property holds with  $B(y, \delta)$  replaced by the interval  $J := B(y, \delta) \cap J(f)$ .

Let  $\mathcal{O}$  be the forward orbit of  $y$ . Note that  $\mathcal{O} \subset \partial I$ , that the set  $\mathcal{O}' := f^{-1}(\mathcal{O}) \cap \partial J(f)$  is forward invariant, and that  $f^{-1}(\mathcal{O}') \setminus \mathcal{O}'$  is contained in the interior of  $J(f)$ . Reducing  $\delta$  if necessary assume that each pull-back of  $B(y, \delta)$  by  $f$  or by  $f^2$  that is disjoint from  $\mathcal{O}'$  is contained in  $J(f)$ . It follows that for every integer  $n \geq 1$ , each pull-back  $W$  of  $B(y, \delta)$  by  $f^n$  that is disjoint from  $\mathcal{O}'$  is contained in  $J(f)$  and therefore coincides with a pull-back of  $J$  by  $f^n$ . By our choice of  $\delta$ , in this case we have  $|W| \leq C\lambda_0^{-n}$ . It remains to consider those pull-backs  $W$  of  $B(y, \delta)$  that intersect  $\mathcal{O}'$ . Since by Lemma 3.4 the periodic point  $y$  satisfies  $\chi_y(f) \geq \ln \lambda$ , reducing  $\delta$  if necessary we can assume that for every integer  $n \geq 1$  and every pull-back  $W$  of  $B(y, \delta)$  by  $f^n$  that intersects  $\mathcal{O}'$ , we have  $|W| \leq C\lambda_0^{-n}$ . This completes the proof of the proposition.  $\square$

#### 4. QUANTIFYING ASYMPTOTIC EXPANSION

The purpose of this section is to prove the following strengthened version of the Main Theorem. Given a compact space  $X$  and a continuous map  $T : X \rightarrow X$ , we denote by  $\mathcal{M}(X, T)$  the space of Borel probability measures on  $X$  that are invariant by  $T$ .

**Main Theorem'.** *For an interval map  $f$  in  $\mathcal{A}$ , the number*

$$\chi_{\inf}(f) := \{\chi_\nu(f) : \nu \in \mathcal{M}(J(f), f)\}$$

*is equal to*

$$\chi_{\text{per}}(f) := \inf \{\chi_p(f) : p \text{ hyperbolic repelling periodic point of } f \text{ in } J(f)\}.$$

If in addition  $f$  is topologically exact on  $J(f)$ , then there is  $\delta' > 0$  such that the following properties hold. Let  $J$  be an interval contained in  $I$  that intersects  $J(f)$  and satisfies  $|J| \leq \delta'$ . If  $\chi_{\inf}(f) > 0$  and  $J(f)$  is not an interval, then assume  $J$  is not a neighborhood of a periodic point in the boundary of a Fatou component of  $f$ . Then:

1. For every  $\chi < \chi_{\inf}(f)$  there is a constant  $C > 0$  independent of  $J$ , such that for every integer  $n \geq 1$  and every pull-back  $W$  of  $J$  by  $f^n$ , we have  $|W| \leq C \exp(-n\chi)$ .
2. We have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \ln \max \{|W| : W \text{ connected component of } f^{-n}(J)\} = -\chi_{\inf}(f).$$

Finally, for each point  $x_0$  in  $J(f)$  we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \ln \min \{|Df^n(x)| : x \in f^{-n}(x_0)\} \leq \chi_{\inf}(f),$$

and there is a subset  $E$  of  $J(f)$  of zero Hausdorff dimension such that for each point  $x_0$  in  $J(f) \setminus E$  the  $\limsup$  above is a limit and the inequality an equality.

*Remark 4.1.* There is an example showing that the hypothesis in the Main Theorem' that  $J$  is not a neighborhood of a periodic point in the boundary of a Fatou component, is necessary, see [RL12, Proposition A]. However, a qualitative result holds when this hypothesis is not satisfied, see [RL12, Theorem B].

The proof of the Main Theorem' is given below, after the following lemmas from [PRLS03].

When  $f$  is a complex rational map the following lemma is a direct consequence of [PRLS03, Lemma 3.1]. Using Fact 2.2, the proof applies without change to the case where  $f$  is a map in  $\mathcal{A}$ .

**Lemma 4.2.** *Let  $f$  be an interval map in  $\mathcal{A}$  that is topologically exact on  $J(f)$  and such that  $\chi_{\text{per}}(f) > 0$ . Then there is a point  $x_0$  in  $J(f)$  such that*

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \min \{|Df^n(x)| : x \in f^{-n}(x_0)\} \geq \chi_{\text{per}}(f).$$

In the case where  $f$  is a complex rational map, the following is [PRLS03, Lemma 2.1 and Remark 2.2]. The proof applies without change to maps in  $\mathcal{A}$ .

**Lemma 4.3.** *Let  $f : I \rightarrow I$  be a map in  $\mathcal{A}$ . Then there are  $\delta_5 > 0$  and a subset  $E$  of  $I$  of zero Hausdorff dimension, such that for every interval  $J$  contained in  $I$  that intersects  $J(f)$  and satisfies  $|J| \leq \delta_5$  and every point  $x_0$  in  $J \setminus E$ , we have*

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \frac{1}{n} \ln \min \{|Df^n(x)| : x \in f^{-n}(x_0)\} \\ \geq - \limsup_{n \rightarrow +\infty} \frac{1}{n} \ln \max \{|W| : W \text{ connected component of } f^{-n}(J)\}. \end{aligned}$$

*Proof of the Main Theorem'.* To prove

$$(4.1) \quad \chi_{\inf}(f) = \chi_{\text{per}}(f),$$

suppose  $f$  is “infinitely renormalizable,” see [dMvS93] for the definition and for precisions. It follows easily from the *a priori* bounds in [vSV04] that in this case we have  $\chi_{\inf}(f) = \chi_{\text{per}}(f) = 0$ . So, to prove (4.1) it is enough to consider the case



where  $f$  is at most finitely renormalizable. Then  $f$  can be decomposed into finitely many interval maps, each of which has a renormalization with a topologically exact restriction, see for example [dMvS93, §III, 4]. Thus, to prove the Main Theorem it is enough to consider the case where  $f$  is topologically exact.

In part 1 below we prove part 1 with  $\chi_{\text{inf}}(f)$  replaced by  $\chi_{\text{per}}(f)$  and in part 2 we prove  $\chi_{\text{per}}(f) = \chi_{\text{inf}}(f)$ . We complete the proof of the theorem in part 3.

**1.** We prove part 1 of the theorem with  $\chi_{\text{inf}}(f)$  replaced by  $\chi_{\text{per}}(f)$ . This statement being trivial in the case where  $\chi_{\text{per}}(f) = 0$ , we suppose  $\chi_{\text{per}}(f) > 0$ . Combining Lemma 4.2 and Proposition 3.1 we obtain that all the periodic points of  $f$  in  $J(f)$  are hyperbolic repelling and that for every  $\chi$  in  $(0, \chi_{\text{per}}(f))$  there is  $\delta(\chi) > 0$  such that for every interval  $J$  that intersects  $J(f)$ , that is disjoint from each periodic Fatou component of  $f$ , and that satisfies  $|J| \leq \delta(\chi)$ , the following property holds: For every integer  $n \geq 1$  and every pull-back  $W$  of  $J$  by  $f^n$  we have

$$|W| \leq \exp(-n\chi).$$

Put  $\delta' := \delta(\chi_{\text{per}}(f)/2)$  and let  $J$  be an interval that intersects  $J(f)$ , that is disjoint from the periodic Fatou components of  $f$ , and that satisfies  $|J| \leq \delta'$ . Given  $\chi$  in  $(\chi_{\text{per}}(f)/2, \chi_{\text{per}}(f))$ , let  $N \geq 1$  be sufficiently large so that  $\exp(-N\chi) \leq \delta(\chi)$ , let  $n \geq N$  be an integer, and let  $W$  be a pull-back of  $J$  by  $f^n$ . If we denote by  $W'$  the pull-back of  $J$  by  $f^N$  containing  $f^{n-N}(W)$ , then we have

$$|W'| \leq \exp(-N\chi) \leq \delta(\chi).$$

So the property above applied to  $W'$  instead of  $J$  implies

$$|W| \leq \exp(-(n-N)\chi).$$

This proves part 1 of the theorem with  $C = \exp(N\chi)$  and with  $\chi_{\text{inf}}(f)$  replaced by  $\chi_{\text{per}}(f)$ .

**2.** We prove  $\chi_{\text{per}}(f) = \chi_{\text{inf}}(f)$ . To prove  $\chi_{\text{per}}(f) \geq \chi_{\text{inf}}(f)$ , let  $p$  be a hyperbolic repelling periodic point of  $f$  in  $J(f)$  and let  $\nu$  be the probability measure equidistributed on the orbit of  $p$ . Then  $\nu$  is in  $\mathcal{M}(J(f), f)$  and  $\chi_\nu(f) = \chi_p(f)$ , so  $\chi_p(f) \geq \chi_{\text{inf}}(f)$ . This proves  $\chi_{\text{per}}(f) \geq \chi_{\text{inf}}(f)$ . To prove the reverse inequality we show that for every  $\nu$  in  $\mathcal{M}(J(f), f)$  we have  $\chi_\nu(f) \geq \chi_{\text{per}}(f)$ . By the ergodic decomposition theorem we can assume without loss of generality that  $\nu$  is ergodic. By [Prz93, Theorem B] or by Proposition A.1 in Appendix A, we have  $\chi_\nu(f) \geq 0$ . We show that for every  $\varepsilon > 0$  there is a point  $x$  in  $J(f)$  such that for every sufficiently large integer  $n \geq 1$  we have

$$(4.2) \quad f^n(B(x, \exp(-(\chi_\nu(f) + 2\varepsilon)n))) \subset B(f^n(x), \exp(-\varepsilon n)).$$

Using this estimate with a sufficiently large  $n$  and combining it with part 1 we obtain  $\chi_\nu(f) + 2\varepsilon \geq \chi_{\text{per}}(f)$ . Since  $\nu$  and  $\varepsilon$  are arbitrary, this proves  $\chi_{\text{inf}}(f) \geq \chi_{\text{per}}(f)$ , as wanted. To prove (4.2), note that by Birkhoff's ergodic theorem there is a point  $x_0$  in  $J(f)$  and an integer  $n_0 \geq 1$  such that for every  $n \geq n_0$  we have

$$(4.3) \quad \exp((\chi_\nu(f) - \frac{1}{3}\varepsilon)n) \leq |Df^n(x_0)| \leq \exp((\chi_\nu(f) + \frac{1}{3}\varepsilon)n).$$

On the other hand, since the critical points of  $f$  are non-flat, there is a constant  $C_0 > 0$  such that for every  $x$  in  $I$  we have  $|Df(x)| \leq C_0 \text{dist}(x, \text{Crit}(f))$ . Using this inequality with  $x = f^n(x_0)$ , combined with

$$Df^{n+1}(x_0) = Df(f^n(x_0)) \cdot Df^n(x_0),$$

with (4.3) and with (4.3) with  $n$  replaced by  $n + 1$ , we obtain that for every  $n \geq n_0$  we have

$$\text{dist}(f^n(x), \text{Crit}(f)) \geq (C_0^{-1} \exp(\chi_\nu(f))) \exp\left(-\frac{2}{3}\varepsilon(n+1)\right).$$

This implies that there is an integer  $n_1 \geq n_0$  such that for every  $n \geq n_1$  the distortion of  $f$  on  $B(f^n(x_0), \exp(-\varepsilon n))$  is bounded by  $\exp(\frac{1}{3}\varepsilon)$ . Let  $n_2 \geq n_1$  be sufficiently large so that the distortion of  $f^{n_1}$  on  $B(x_0, \exp(-(\chi_\nu(f) + \varepsilon)n_2))$  is bounded by  $\exp(\frac{1}{3}\varepsilon n_1)$ . Then for every  $n \geq n_2$  we have,

$$(4.4) \quad f^{n_1}(B(x_0, \exp(-(\chi_\nu(f) + 2\varepsilon)n))) \\ \subset B(f^{n_1}(x_0), \exp(-(\chi_\nu(f) + 2\varepsilon)n + \frac{1}{3}\varepsilon n_1)) |Df^{n_1}(x_0)|).$$

Fix  $n \geq n_2$ . We prove by induction that for every  $j$  in  $\{n_1, \dots, n\}$  the inclusion above holds with  $n_1$  replaced by  $j$ . The desired assertion is obtained from this with  $j = n$ , combined with (4.3). Noting that the case  $j = n_1$  is given by (4.4) itself, let  $j$  in  $\{n_1, \dots, n-1\}$  be given and suppose (4.4) holds with  $n_1$  replaced by  $j$ . Then (4.4) with  $n_1$  replaced by  $j+1$  is obtained by using that the right hand side of (4.4) with  $n_1$  replaced by  $j$  is contained in  $B(f^j(x_0), \exp(-\varepsilon n))$ , combined with the fact that the distortion of  $f$  on this last set is bounded by  $\exp(\frac{1}{3}\varepsilon)$ . This completes the proof of the induction step, and hence that  $\chi_\nu(f) \geq \chi_{\text{per}}(f)$  and  $\chi_{\text{inf}}(f) = \chi_{\text{per}}(f)$ .

**3.** So far we have shown part 1 of the theorem and the equality  $\chi_{\text{inf}}(f) = \chi_{\text{per}}(f)$ . Let  $\chi_{\text{per}}^0(f)$  be as in the statement of Lemma 3.4. Clearly,

$$\chi_{\text{inf}}(f) \leq \chi_{\text{per}}^0(f) \leq \chi_{\text{per}}(f)$$

(cf., first part of part 2), so  $\chi_{\text{per}}^0(f) = \chi_{\text{inf}}(f)$ . Thus, inequality (3.2) of Lemma 3.4 and part 1 of the theorem imply part 2 of the theorem. In turn, part 2 of the theorem together with (3.3) of Lemma 3.4 and with Lemma 4.3 imply the last assertion of the theorem. The proof of the theorem is thus complete.  $\square$

## 5. CONJUGACY TO A PIECEWISE AFFINE MAP

In this section we show that a conjugacy between 2 Lipschitz continuous multimodal maps that satisfy the Exponential Shrinking of Components condition<sup>||</sup> is bi-Hölder continuous (Proposition 5.2). Combined with Lemma 5.1 below, this proves implication 5  $\Rightarrow$  4 of Corollary A.

A multimodal map  $f$  is *expanding*, if there is  $\lambda > 1$  so that for every  $x$  and  $x'$  contained in an interval on which  $f$  is monotonous, we have

$$|f(x) - f(x')| \geq \lambda|x - x'|.$$

In this case we say  $\lambda$  is an *expansion constant* of  $f$ .

**Lemma 5.1.** *Every expanding multimodal map satisfies the Exponential Shrinking of Components condition.*

In this section, a turning point  $c$  of a multimodal map  $f$  is *exposed* if for every integer  $n \geq 1$  the point  $f^n(c)$  is not a turning point of  $f$ .

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<sup>||</sup>The Exponential Shrinking of Components condition is defined in §1.2 for non-degenerate smooth interval maps. In this section we apply this definition to multimodal maps.

*Proof.* Let  $f : I \rightarrow I$  be an expanding multimodal map and let  $\lambda > 1$  be an expansion constant of  $f$ . Let  $L \geq 1$  be a sufficiently large integer so that  $\lambda^L > 2$  and let  $\delta_{\dagger} > 0$  be sufficiently small so that for every exposed turning point  $c$  of  $f$  and every  $j$  in  $\{1, \dots, L\}$  the set  $f^j(B(c, \delta_{\dagger}))$  does not contain a turning point of  $f$ . Let  $\delta_* > 0$  be sufficiently small so that for every interval  $J$  contained in  $I$  that satisfies  $|J| \leq \delta_*$  and every connected component  $W$  of  $f^{-1}(J)$  we have  $|W| \leq \delta_{\dagger}$ .

We prove by induction in  $n \geq 0$  that for every interval  $J$  contained in  $I$  that satisfies  $|J| \leq \delta_*/2$ , every  $j$  in  $\{1, \dots, n\}$ , and every pull-back  $W$  of  $J$  by  $f^j$  we have

$$|W| \leq \left(2^{\frac{1}{L}} \lambda^{-1}\right)^n \delta_*.$$

This implies that  $f$  satisfies the Exponential Shrinking of Components condition. The case  $n = 0$  being trivial, suppose that for some  $n \geq 1$  this assertion holds with  $n$  replaced by each element of  $\{0, \dots, n-1\}$ . Let  $J$  be an interval contained in  $I$  that satisfies  $|J| \leq \delta_*/2$  and let  $W$  be a pull-back of  $J$  by  $f^n$ . The induction hypothesis implies for every  $j$  in  $\{1, \dots, n-1\}$  we have  $|f^j(W)| \leq \delta_*$ . Using the hypothesis  $|J| \leq \delta_*/2$  and the definition of  $\delta_*$ , we conclude that for every  $i$  in  $\{0, \dots, n-1\}$  we have  $|f^i(W)| \leq \delta_{\dagger}$ . Using the definition of  $\delta_{\dagger}$ , this implies that the number of those  $i$  in  $\{0, \dots, n-1\}$  such that  $f^i(W)$  contains a turning point of  $f$  in its interior is at most  $\frac{n}{L} + 1$ . It thus follows that  $W$  can be partitioned into at most  $2^{\frac{n}{L}+1}$  intervals on each of which  $f^n$  is injective. Using that  $\lambda$  is an expansion constant of  $f$ , we obtain

$$|W| \leq 2^{\frac{n}{L}+1} \lambda^{-n} |J| \leq 2^{\frac{n}{L}} \lambda^{-n} \delta_*.$$

This completes the proof of the induction hypothesis and of the lemma.  $\square$

**Proposition 5.2.** *Let  $f : I \rightarrow I$  be a Lipschitz continuous multimodal map and  $\tilde{f} : \tilde{I} \rightarrow \tilde{I}$  a multimodal map satisfying the Exponential Shrinking of Components condition. If  $h : I \rightarrow \tilde{I}$  is a homeomorphism conjugating  $f$  to  $\tilde{f}$ , then  $h$  is Hölder continuous.*

We deduce this proposition as an easy consequence of the following lemma.

**Lemma 5.3.** *Let  $f : I \rightarrow I$  be a multimodal map satisfying the Exponential Shrinking of Components condition with constant  $\lambda > 1$ . Then for every  $A > (\ln \lambda)^{-1}$  there is a constant  $\delta_6 > 0$  such that for every interval  $J$  contained in  $I$  the following property holds: There is an integer  $m \geq 0$  that satisfies  $m \leq \max\{-A \ln |J|, 0\}$  and an interval  $J_0$  contained in  $J$ , such that  $f^m$  is injective on  $J_0$  and  $|f^m(J_0)| \geq \delta_6$ .*

*Proof.* Put  $\chi := \ln \lambda$  and let  $L$  be an integer satisfying  $L > (A\chi - 1)^{-1} A \ln 2$ . Let  $\delta_{\dagger} > 0$  be sufficiently small so that for every exposed turning point  $c$  of  $f$  and for every  $j$  in  $\{1, \dots, L\}$ , the set  $f^j(B(c, \delta_{\dagger}))$  does not contain a turning point of  $f$ . Let  $\delta_{\text{Exp}} > 0$  be the constant  $\delta$  given by the Exponential Shrinking of Components condition, see §1.2. Reducing  $\delta_{\text{Exp}}$  if necessary we assume that for every interval  $J$  contained in  $I$  that satisfies  $|J| \leq \delta_{\text{Exp}}$ , every integer  $n \geq 1$ , and every pull-back  $W$  of  $J$  by  $f^n$  we have  $|W| \leq \delta_{\dagger}$ . Let  $\delta_{\text{Exp}}^* > 0$  be such that for every interval  $J$  contained in  $I$  that satisfies  $|J| \geq \delta_{\text{Exp}}$  and for every connected component  $W$  of  $f^{-1}(J)$  we have  $|W| \geq \delta_{\text{Exp}}^*$ . Reducing  $\delta_{\text{Exp}}^*$  if necessary we assume  $\delta_{\text{Exp}}^* \leq \delta_{\text{Exp}}$ . Observing that  $1 + A \frac{\ln 2}{L} < \chi A$ , it follows that there is  $n_0 \geq 1$  such that for every

integer  $n \geq n_0$  we have,

$$(5.1) \quad -A \ln \frac{\delta_{\text{Exp}}^*}{2} + \left(1 + A \frac{\ln 2}{L}\right) n \leq \chi A n.$$

In part 1 below we show that every interval contains an interval that is mapped bijectively by an iterate of  $f$  onto a relatively large interval. In part 2 we use this fact to prove the lemma by induction.

**1.** We prove that for every integer  $n \geq 1$  and every interval  $J$  contained in  $I$  that satisfies  $|J| \geq \exp(-(n+1)\chi)$ , there is  $m$  in  $\{0, \dots, n\}$  and an interval  $J_0$  contained in  $J$  such that  $f^m$  is injective on  $J_0$  and

$$|f^m(J_0)| \geq \frac{\delta_{\text{Exp}}^*}{2} 2^{-\frac{m}{L}}.$$

If  $|J| \geq \delta_{\text{Exp}}$ , then the assertion follows with  $J_0 = J$  and  $m = 0$  from our assumption that  $\delta_{\text{Exp}} \geq \delta_{\text{Exp}}^*$ . Assume  $|J| \leq \delta_{\text{Exp}}$  and note that by the Exponential Shrinking of Components condition, for every integer  $m \geq n+1$  we have  $|f^m(J)| > \delta_{\text{Exp}}$ . So there is a largest integer  $m \geq 0$  such that  $|f^m(J)| \leq \delta_{\text{Exp}}$  and  $m$  satisfies  $m \leq n$ . By definition of  $\delta_{\text{Exp}}^*$  we have  $|f^m(J)| \geq \delta_{\text{Exp}}^*$ . On the other hand, by our choice of  $\delta_{\text{Exp}}$ , for every  $j$  in  $\{0, \dots, m-1\}$  we have  $|f^j(J)| \leq \delta_{\dagger}$ . From the definition of  $\delta_{\dagger}$  it follows that the number of those  $j$  in  $\{0, \dots, m-1\}$  such that  $f^j(J)$  contains a turning point in its interior is bounded by  $\frac{m}{L} + 1$ . This implies that  $J$  can be partitioned into at most  $2^{\frac{m}{L}+1}$  intervals on which  $f^m$  is injective. So, if we denote by  $J_0$  an interval  $J'$  in this partition for which  $|f^m(J')|$  is maximal, then we have

$$(5.2) \quad |f^m(J_0)| \geq \frac{|f^m(J)|}{2^{\frac{m}{L}+1}} \geq \frac{\delta_{\text{Exp}}^*}{2} 2^{-\frac{m}{L}}.$$

**2.** Put  $\delta_6 := \frac{\delta_{\text{Exp}}^*}{2} 2^{-\frac{n_0}{L}}$ . We prove by induction that for every integer  $n \geq 1$  the lemma holds for every interval  $J$  that satisfies  $|J| \geq \exp(-(n+1)\chi)$ . Part 1 implies that this holds for every integer  $n \geq 0$  satisfying  $n \leq n_0$ . Let  $n \geq n_0$  be an integer for which the lemma holds for every interval  $J$  that satisfies  $|J| \geq \exp(-n\chi)$ . To prove the inductive step, let  $J$  be a given interval contained in  $I$  that satisfies

$$\exp(-(n+1)\chi) \leq |J| \leq \exp(-n\chi).$$

Let  $m$  be the integer in  $\{0, \dots, n\}$  and  $J_0$  the interval contained in  $J$  given by part 1. So  $f^m$  is injective on  $J_0$  and

$$|f^m(J_0)| \geq \frac{\delta_{\text{Exp}}^*}{2} 2^{-\frac{m}{L}} \geq \frac{\delta_{\text{Exp}}^*}{2} 2^{-\frac{n}{L}}.$$

Together with (5.1) this implies  $|f^m(J_0)| \geq \exp(-n\chi)$ , so we can apply the induction hypothesis with  $J$  replaced by  $f^m(J_0)$ . Therefore there is an interval  $J'_0$  contained in  $f^m(J_0)$  and an integer  $m' \geq 0$  satisfying  $m' \leq \max\{-A \ln |f^m(J_0)|, 0\}$ , such that  $f^{m'}$  is injective on  $J'_0$  and  $|f^{m'}(J'_0)| \geq \delta_6$ . If  $m' = 0$ , then  $|f^m(J_0)| \geq |J'_0| \geq \delta_6$ . Together with

$$m \leq n \leq -\chi^{-1} \ln |J| < -A \ln |J|,$$

this completes the proof of the induction step in the case where  $m' = 0$ . Suppose  $m' \geq 1$  and let  $\tilde{J}_0$  be the connected component of  $f^{-m}(J'_0)$  contained in  $J_0$ ,

so that  $f^m$  is injective on  $\tilde{J}_0$  and  $f^m(\tilde{J}_0) = J'_0$ . Then  $f^{m+m'}$  is injective on  $\tilde{J}_0$  and  $|f^{m+m'}(\tilde{J}_0)| = |f^{m'}(J'_0)| \geq \delta_6$ . On the other hand, we have by (5.1) and (5.2)

$$\begin{aligned} m + m' &\leq m - A \ln |f^m(J_0)| \leq -A \ln \frac{\delta_{\text{Exp}}^*}{2} + \left(1 + A \frac{\ln 2}{L}\right) m \\ &\leq \chi A n \leq -A \ln |J|. \end{aligned}$$

This completes the proof of the induction step with  $m$  replaced by  $m+m'$  and  $J_0$  replaced by  $\tilde{J}_0$ . The proof of the lemma is thus complete.  $\square$

*Proof of Proposition 5.2.* Denote by  $M$  a Lipschitz constant of  $f$ , let  $A$  and  $\delta_6$  be as in Lemma 5.3 with  $f$  replaced by  $\tilde{f}$  and let  $\delta_6^* > 0$  be such that for every interval  $J^*$  contained in  $\tilde{I}$  that satisfies  $|J^*| \geq \delta_6$ , we have  $|h^{-1}(J^*)| \geq \delta_6^*$ .

To prove that  $h$  is Hölder continuous, let  $J$  be an interval contained in  $I$  and let  $m \geq 0$  be the integer and  $J_0$  the interval given by Lemma 5.3 with  $J$  replaced by  $h(J)$ , so that

$$m \leq \max\{-A \ln |h(J)|, 0\}, J_0 \subset h(J), |\tilde{f}^m(J_0)| \geq \delta_6,$$

and so that  $\tilde{f}^m$  is injective on  $J_0$ . It follows that  $f^m$  is injective on  $h^{-1}(J_0)$ , so by the definition of  $\delta_6^*$  we have

$$|J| \geq |h^{-1}(J_0)| \geq M^{-m} |h^{-1}(\tilde{f}^m(J_0))| \geq \min\{|h(J)|^{A \ln M}, 1\} \cdot \delta_6^*.$$

This proves that  $h$  is Hölder continuous of exponent  $(A \ln M)^{-1}$ .  $\square$

## 6. NON-UNIFORM HYPERBOLICITY CONDITIONS

The purpose of this section is to prove Corollaries A and E.

*Proof of Corollary A.* To prove that conditions 1–6 are equivalent, remark first that the equivalence between conditions 1, 2, 5 and 6 is given by the Main Theorem', using Fact 2.2 for the implication  $5 \Rightarrow 6$ . When  $f$  is a complex rational map, the implication  $5 \Rightarrow 3$  is [PRL07, Theorem C]. The proof applies without change to the case where  $f$  is a non-degenerate smooth interval map that is topologically exact.\*\* When  $f$  is unimodal, the implication  $3 \Rightarrow 2$  is [NS98, Lemma 8.2]. The proof applies without change to the general case. We complete the proof that conditions 1–6 are equivalent by showing the implications  $5 \Rightarrow 4$  and  $4 \Rightarrow 2$ . For the implication  $5 \Rightarrow 4$ , recall that by the general theory of Parry [Par66] and of Milnor and Thurston [MT88], the map  $f$  is conjugated to a piecewise affine expanding map. That the conjugacy is bi-Hölder follows from the combination of Lemma 5.1 and Proposition 5.2. When  $f$  is unimodal, the implication  $4 \Rightarrow 2$  is [NS98, Lemma 8.4]. The proof applies without change to the general case. This completes the proof that conditions 1–6 are equivalent.

To prove the final statement, note that the Backward Collet-Eckmann condition implies condition 6 trivially. On the other hand, the Collet-Eckmann condition implies condition 2 by [BvS03, Corollary 1.1].  $\square$

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\*\*For a proof written for maps in  $\mathcal{A}$ , see [RLS10, Remark 2.14]. If in addition  $f$  satisfies Collet-Eckmann condition and  $J(f) = I$ , see also [KN92, You92] if  $f$  is unimodal, [BLVS03] if all the critical points of  $f$  are of the same order and [GS09, Theorem 6] if  $f$  is real analytic.

*Remark 6.1.* Conditions 1, 2, 5 and 6 of Corollary A have natural formulations for maps in  $\mathcal{A}$ . The Main Theorem' implies these conditions are equivalent, using Fact 2.2 for the implication  $5 \Rightarrow 6$ . Using conformal measures, a condition analogous to condition 3 of Corollary A can also be stated for a general interval map  $f$  in  $\mathcal{A}$ . Our results imply that in this more general setting condition 3 is equivalent to conditions 1, 2, 5 and 6. In fact, the implication  $5 \Rightarrow 3$  is again given by either [PRL07, Theorem C] or [RLS10, Remark 2.14]. The proof of the implication  $3 \Rightarrow 2$  for unimodal maps in [NS98, Lemma 8.2] does not apply directly to this more general setting, as it uses that the reference measure is the Lebesgue measure. Using Frostman's lemma, the argument can be adapted to deal with the case where the reference measure is a conformal measure, as in [PRL07, Theorem D] for complex rational maps.

*Remark 6.2.* Condition 4 of Corollary A can be formulated in terms of the maximal entropy measure, as follows. Let  $f : I \rightarrow I$  be a non-degenerate smooth map that is topologically exact. First notice that the conjugacy  $h : I \rightarrow [0, 1]$  to the piecewise affine model is Hölder continuous by Lemma 5.1 and Proposition 5.2. Thus condition 4 is equivalent to the condition that  $h^{-1}$  is Hölder continuous. The conjugacy  $h$  is defined in terms of the maximal entropy measure  $\rho_f$  of  $f$  as follows: If we denote by  $a$  the left end point of  $I$ , then for every  $x$  in  $I$  we have  $h(x) = \rho_f([a, x])$ . Thus, condition 4 is equivalent to the existence of constants  $C > 0$  and  $\alpha > 0$ , such that for every interval  $J$  contained in  $I$  we have  $\rho_f(J) \geq C|J|^\alpha$ .

When  $f$  is a complex rational map, the analogous property of the maximal entropy measure is equivalent to the TCE condition [RL10, Theorem B]. Compare with [PRLS03], where for a complex rational map  $f$  condition 4 was interpreted as the existence of "Hölder coding tree."

*Remark 6.3.* Both, the Collet-Eckmann and the Backward Collet-Eckmann condition have natural formulations for maps in  $\mathcal{A}$ . In this more general setting each of these conditions implies conditions 1–3, 5, and 6 of Corollary A, see Remark 6.1. In fact, the Backward Collet-Eckmann condition implies condition 6 trivially and the Collet-Eckmann condition implies condition 2 by [BvS03, Corollary 1.1]. We note also that for a map in  $\mathcal{A}$  the Collet-Eckmann condition implies the Backward Collet-Eckmann condition at each critical point of maximal order: for complex rational maps this is given by [GS98, Theorem 1]; the proof applies without change to maps in  $\mathcal{A}$ .<sup>††</sup>

*Proof of Corollary D.* We show that for a non-degenerate smooth map  $f : I \rightarrow I$ , an iterate of  $f$  has an exponentially mixing acip, if and only if:

- (\*) There is an interval  $J$  contained in  $I$  and an integer  $s \geq 1$ , such that  $f^s(J) \subset J$  and such that  $f^s : J \rightarrow J$  is a topologically exact map that satisfies the TCE condition.

Since (\*) is clearly invariant under topological conjugacy, this implies the corollary.

If (\*) is satisfied, then  $f^s|_J$  is non-injective and therefore it is a non-degenerate smooth interval map. Then Corollary C implies that  $f^s|_J$ , and hence that  $f^s$ , has an exponentially mixing acip.

Suppose there is an integer  $s \geq 1$  such that  $f^s$  has an exponentially mixing acip  $\nu$ , and denote by  $J$  the support of  $\nu$ . Then  $J$  is an interval,  $f^s(J) \subset J$ ,

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<sup>††</sup>In fact, the proof for maps  $\mathcal{A}$  is slightly simpler, as the arguments involving shrinking neighborhoods can be replaced by the one-sided Koebe principle.

and  $f^s|_J$  is topologically exact, see [vSV04, Theorem E (2)]. It follows that  $f^s_J$  is non-injective and therefore that  $f^s|_J$  is a non-degenerate smooth interval map. Thus Corollary C implies that  $f^s|_J$  satisfies the TCE condition. This proves that  $f$  satisfies (\*), and completes the proof of the corollary.  $\square$

*Remark 6.4.* The proof of Corollary D applies without change to maps in  $\mathcal{A}$ .

*Proof of Corollary E.* Denote by  $I$  the domain of  $f$ .

The implications  $2 \Rightarrow 1$  and  $2 \Rightarrow 3$  are trivial and the implication  $3 \Rightarrow 2$  follows from the fact that the geometric pressure function  $P$  is non-increasing. To prove the implication  $3 \Rightarrow 4$ , suppose 3 holds, let  $\chi > 0$  be given, and let  $\nu$  be a measure in  $\mathcal{M}(I, f)$  such that  $h_\nu(f) - 2\chi_\nu(f) \geq -\chi$ . By [Prz93, Theorem B] or Proposition A.1, we have  $\chi_\nu(f) \geq 0$ . Combined with Ruelle's inequality  $h_\nu(f) \leq \chi_\nu(f)$  [Rue78], we obtain

$$2\chi_\nu(f) \leq h_\nu(f) + \chi \leq \chi_\nu(f) + \chi \text{ and } \chi_\nu(f) \leq \chi.$$

Since  $\chi$  is arbitrary, this shows that  $\chi_{\inf}(f) = 0$  and therefore that  $f$  does not satisfy the TCE condition by Corollary C. To prove the implication  $4 \Rightarrow 3$ , suppose that  $f$  does not satisfy the TCE condition, let  $t_0 > 0$  be the first zero of  $P$ , and let  $t > t_0$  and  $\chi > 0$  be given. By Corollary C we have  $\chi_{\inf}(f) = 0$ , so there is a measure  $\nu$  in  $\mathcal{M}(I, f)$  such that  $\chi_\nu(f) < \chi$ . So,

$$P(t) \geq h_\nu(f) - t\chi_\nu(f) \geq -t\chi.$$

Since  $\chi > 0$  is arbitrary we conclude that  $P(t) \geq 0$  and hence that  $P$  is non-negative.

We complete the proof of the corollary by showing the implication  $1 \Rightarrow 4$ . Suppose  $f$  has a high-temperature phase transition. In [PRL12] it is shown that  $P$  is real analytic until its first zero, so  $f$  has a phase transition at the first zero of  $P$ . However, in [PRL12] it is also shown that for a map satisfying the TCE condition the pressure function is real analytic at its first zero. We thus conclude that  $f$  does not satisfy the TCE condition. This completes the proof of the implication  $1 \Rightarrow 4$  and of the corollary.  $\square$

*Remark 6.5.* Each of the conditions 1–4 of Corollary E have natural formulations in the case where  $f$  is an interval map in  $\mathcal{A}$ . The proof of Corollary E applies without change in this more general setting.

## APPENDIX A. LYAPUNOV EXPONENTS ARE NON-NEGATIVE

In this appendix we prove the following general result characterizing those invariant measures whose Lyapunov exponent is strictly negative (possibly infinite). For smooth interval maps with a finite number of non-flat critical points, this was shown by Przytycki in [Prz93, Theorem B]. We give a proof of this important fact that avoids the Koebe principle and applies to continuously differentiable maps. It is considerably shorter than the proof in [Prz93] and extends without change to complex rational maps.

For a continuously differentiable interval map  $f$ , a periodic point  $p$  of period  $n$  of  $f$  is *hyperbolic attracting* if  $|Df^n(p)| < 1$ . For a Borel measure  $\nu$  on a topological space  $X$ , we use  $\text{supp}(\nu)$  to denote the support of  $\nu$ , which is by definition the set of all points in  $X$  such that the measure of each of its neighborhoods is strictly positive.

**Proposition A.1.** *Let  $f$  be a continuously differentiable interval map and let  $\nu$  be an ergodic invariant probability measure. Then either  $\chi_\nu(f) \geq 0$  or  $\nu$  is supported on a hyperbolic attracting periodic orbit of  $f$ .*

*Proof.* Suppose  $\chi_\nu(f) < 0$ . By the dominated convergence theorem there exists  $L > 0$  such that the function

$$\varphi := \max\{\ln|Df|, -L\}$$

satisfies  $A := \int \varphi d\nu < 0$ . Fix  $\chi$  in  $(0, -A/3)$  and for each integer  $n \geq 1$  put

$$S_n(\varphi) := \varphi + \varphi \circ f + \cdots + \varphi \circ f^{n-1}.$$

1. We show that for every point  $x$  in the domain  $I$  of  $f$  satisfying

$$\lim_{n \rightarrow +\infty} \frac{1}{n} S_n(\varphi)(x) = A,$$

there exists  $\tau > 0$  such that for every sufficiently large integer  $n$  we have  $|Df^n| \leq \exp(-\chi n)$  on  $B(x, \tau)$ . Fix such  $x$  in  $I$  and let  $\delta > 0$  be such that we have  $|Df| \leq \exp(-L)$  on  $B(\text{Crit}(f), \delta)$ . As  $f$  is continuously differentiable there is  $\varepsilon$  in  $(0, \delta/3)$  such that the distortion of  $f$  on an interval of length at most  $\varepsilon$  and disjoint from  $B(\text{Crit}(f), \delta/3)$  is at most  $\exp(\chi)$ . By our choice of  $\chi$  there is  $\tau > 0$  so that for every  $n \geq 0$  we have

$$\tau \exp(S_n(\varphi)(x) + 3n\chi) < \varepsilon/2.$$

Finally, for each  $n \geq 0$  put

$$r_n := \tau \exp(S_n(\varphi)(x) + n\chi) \text{ and } B_n := B(f^n(x), r_n).$$

Note that we have  $|B_n| = 2r_n \leq \varepsilon \exp(-2n\chi)$ .

We show that for every  $n \geq 0$  we have  $|Df| \leq \exp(\varphi(f^n(x)) + \chi)$  on  $B_n$ . This implies that  $f(B_n) \subset B_{n+1}$  and by induction that on  $B(x, \tau)$  we have

$$|Df^n| \leq \exp(S_n(\varphi)(x) + \chi n) \leq \tau^{-1}(\varepsilon/2) \exp(-2n\chi).$$

It then follows that for large  $n$  we have  $|Df^n| \leq \exp(-\chi n)$  on  $B(x, \tau)$ , as wanted.

**Case 1.**  $f^n(x) \notin B(\text{Crit}(f), 2\delta/3)$ . Since the length of  $B_n$  is less than  $\varepsilon < \delta/3$ , it follows that the interval  $B_n$  is disjoint from  $B(\text{Crit}(f), \delta/3)$  and that the distortion of  $f$  on  $B_n$  is bounded by  $\exp(\chi)$ . So on  $B_n$  we have

$$|Df| \leq |Df(f^n(x))| \exp(\chi) \leq \exp(\varphi(f^n(x)) + \chi).$$

**Case 2.**  $f^n(x) \in B(\text{Crit}(f), 2\delta/3)$ . Then  $B_n \subset B(\text{Crit}(f), \delta)$  and by our choice of  $\delta$  we have  $|Df| \leq \exp(-L)$  on  $B_n$ .

2. By Birkhoff's ergodic theorem the set of points  $x$  satisfying the property described in part 1 has full measure with respect to  $\nu$ . We can thus find such a point  $x$  in  $\text{supp}(\nu)$ , such that in addition its orbit is dense in  $\text{supp}(\nu)$ . Let  $\tau > 0$  be given by the property described in part 1 for this choice of  $x$ . Then there is an integer  $n \geq 1$  such that  $|Df^n| \leq \exp(-n\chi) \leq \frac{1}{4}$  on  $B(x, \tau)$  and such that  $f^n(x)$  is in  $B(x, \tau/4)$ . Then

$$f^n(B(x, \tau)) \subset f^n(B(x, \tau/2))$$

and  $f^n$  is uniformly contracting on  $B(x, \tau)$ . This implies that  $x$  is asymptotic to an attracting periodic point of  $f$ . Since  $x$  is in  $\text{supp}(\nu)$  and  $\nu$  is ergodic, it follows that  $\nu$  is supported on a hyperbolic attracting periodic orbit of  $f$ .  $\square$



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